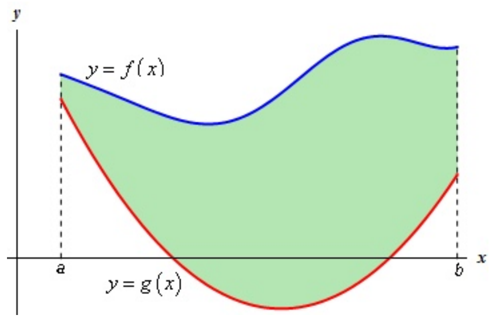
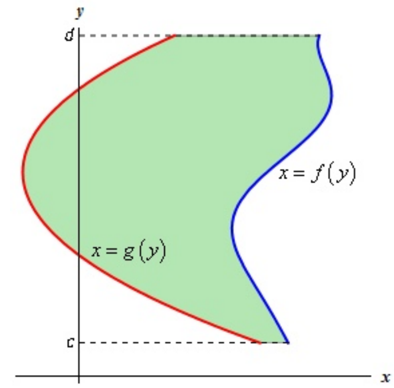
Area between Curves:



Situation ①



Situation ②

We want to find the area between the "Blue Curve" & the "Red Curve" on an interval.

Situation ①

Area is enclosed by the "Blue Curve" — $y = f(x)$

the "Red Curve" — $y = g(x)$

$x = a$ & $x = b$ lines.

Note: Throughout $[a, b]$, $f(x) \geq g(x)$.

So, $$\text{Area} = \int_{x=a}^{x=b} [f(x) - g(x)]\, dx$$

or simply

$$\int_a^b [f(x) - g(x)]\, dx$$

Situation ②

Area is enclosed by the "Blue Curve" — $x = f(y)$

the "Red Curve" — $x = g(y)$

$y = c$ & $y = d$ lines.

Note: Throughout $[c, d]$, $f(y) \geq g(y)$.

So, $\text{Area} = \int_{y=c}^{y=d} [f(y) - g(y)]\, dy$

or simply

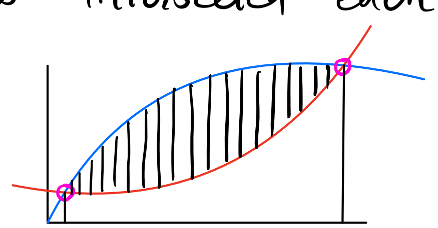
$$\int_c^d [f(y) - g(y)]\, dy$$

General Rule:

① $\text{Area} = \int_a^b (\text{upper function}) - (\text{lower function})\, dx$, $a \leq x \leq b$

② $\text{Area} = \int_c^d (\text{right function}) - (\text{left function})\, dy$, $c \leq y \leq d$.

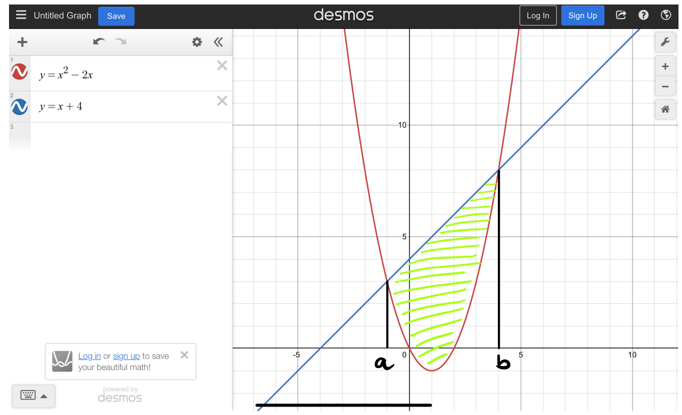
③ For enclosed curves, the interval won't be given. So we need to find all those points where both curves intersected each other.



Q.1 Find the area between the line $y = x+4$ & $y = x^2 - 2x$

parabola.

Sol^n:- First we need a clear understanding of the graph & find the interval over which we are going to find the area.



Note:- Since interval is not given it's indicating for enclosed areas (in light green).

First observation, in the enclosed area the st. Line $y=x+4$ is always on top than the parabola $y = x^2 - 2x$.

So the enclosed area $= \int_{a}^{b} [(x+4) - (x^2 - 2x)]\,dx$.

b unknown

a unknown

To get a & b (ie, the points of intersections), we put

$$x+4 = x^2 - 2x$$

$$\Rightarrow x^2 - 3x - 4 = 0$$

$$\Rightarrow (x+1)(x-4) = 0$$

$$\Rightarrow x = -1 \;,\; x = 4$$

a b

Hence the enclosed area is

$$\int_{-1}^{4}[(x+4)-(x^2-2x)]\,dx = \int_{-1}^{4}[x+4-x^2+2x]\,dx$$

$$= \int_{-1}^{4}(-x^2+3x+4)\,dx$$

$$= \left[-\frac{x^3}{3}+3\cdot\frac{x^2}{2}+4\cdot x\right]_{-1}^{4}$$

$$= \left(-\frac{4^3}{3}+3\cdot\frac{4^2}{2}+4\cdot 4\right)-\left(-\frac{(-1)^3}{3}+3\cdot\frac{(-1)^2}{2}+4(-1)\right)$$

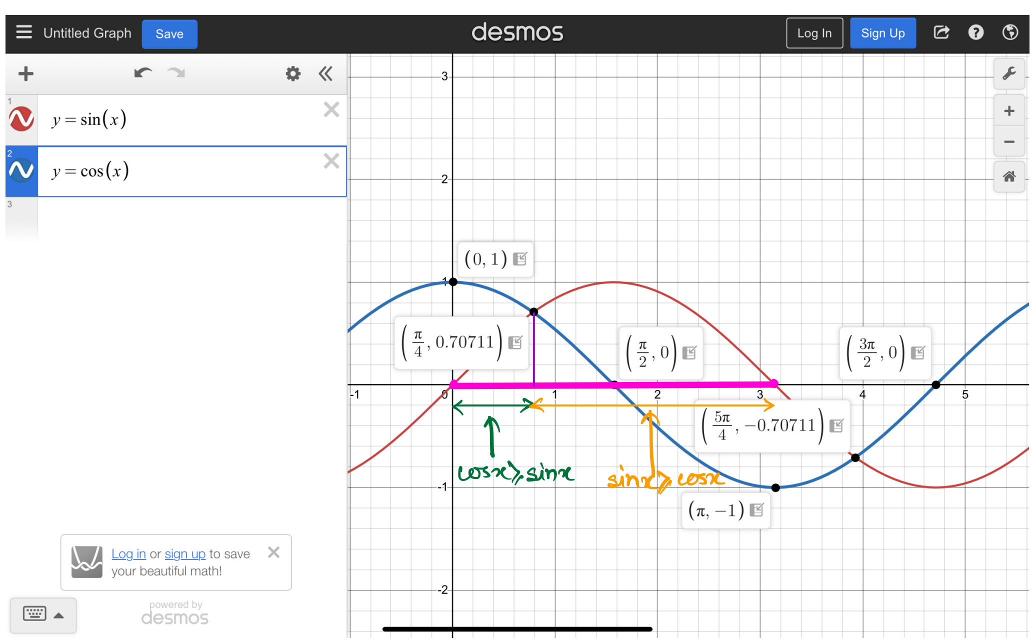
$$= \left(-\frac{64}{3}+3(8)+16\right)-\left(\frac{1}{3}+\frac{3}{2}-4\right)$$

$$= \left(-\frac{65}{3}+\frac{85}{2}\right)$$

$$= \frac{125}{6}$$

Q.2. Find the area between $y=\cos x$ & $y=\sin x$ on $[0,\pi]$.

Sol^n :



$$\text{Area} = \text{Area}(\cos x \geq \sin x) + \text{Area}(\sin x \geq \cos x)$$

$$= \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi} (\sin x - \cos x)\,dx$$

$$= \left[\sin x + \cos x\right]_0^{\pi/4} + \left[-\cos x - \sin x\right]_{\pi/4}^{\pi}$$

$$= \left[\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - (\sin 0 + \cos 0)\right] + \left[\left(-\cos\pi - \sin\pi\right) - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right)\right]$$

$$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1)\right] + \left[(-(-1) - 0) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right]$$

$$= \left(\frac{2}{\sqrt{2}} - 1\right) + \left[1 + \frac{2}{\sqrt{2}}\right]$$

$$= \frac{2}{\sqrt{2}} \cancel{-1} \cancel{+1} + \frac{2}{\sqrt{2}} = 2\left(\frac{2}{\sqrt{2}}\right) = 2\left(\frac{(\sqrt{2})^2}{\sqrt{2}}\right) = 2\sqrt{2}.$$

Q.3. Determine the area enclosed by $x = \frac{y^2}{2} - 3$ & $y = x - 1$.

Sol^n: First we need the point of intersection. So we equate $x = \frac{y^2}{2} - 3$ & $y = x - 1 \Leftrightarrow x = y + 1$.

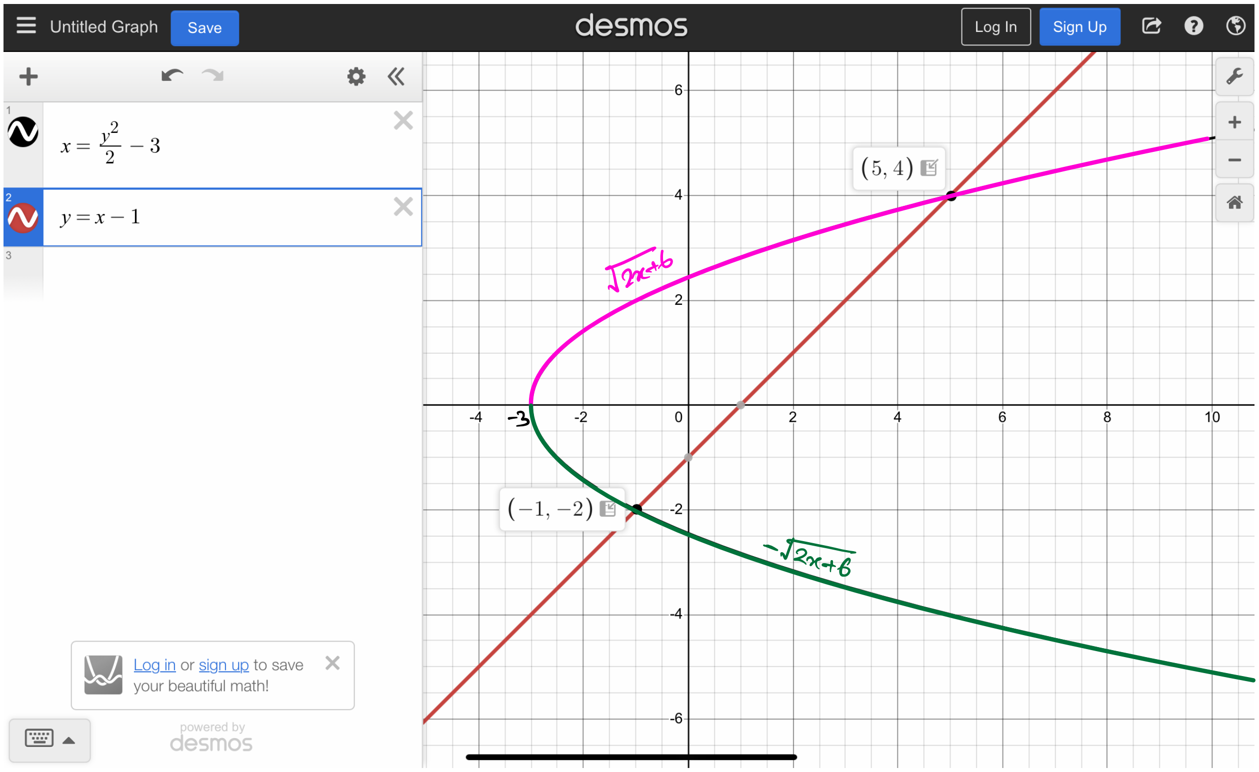
$$\frac{y^2}{2} - 3 = y + 1$$

$$\Rightarrow (y-4)(y+2) = 0 \Rightarrow y = 4 \ \& \ y = -2$$

Now, when $y = 4$, $x = 5 \rightsquigarrow (5, 4)$

&

when $y = -2$, $x = -1 \rightsquigarrow (-1, -2)$



Method ① ~ Over the x-axis. (i.e, integration with dx)

So, $x = \frac{y^2}{2} - 3$

$\Rightarrow 2x = y^2 - 6$

$\Rightarrow 2x + 6 = y^2$

$\Rightarrow y^2 = 2x + 6 \Rightarrow y = \pm\sqrt{2x+6}$

$$\text{Enclosed Area} = \int_{-3}^{-1} \left(\sqrt{2x+6} - \left(-\sqrt{2x+6}\right)\right) dx + \int_{-1}^{5} \left(\sqrt{2x+6} - (x-1)\right) dx$$

$$= \int_{-3}^{-1} 2\sqrt{2x+6}\, dx + \int_{-1}^{5} \sqrt{2x+6}\, dx - \int_{-1}^{5} (x-1)\, dx$$

$$= 2\int_{-3}^{-1} \sqrt{2x+6}\, dx + \int_{-1}^{5} \sqrt{2x+6}\, dx - \left[\frac{x^2}{2} - x\right]_{-1}^{5}$$

Take $u^2 = 2x+6$, then $2u\,du = 2\,dx \Rightarrow u\,du = dx$

x	-3	-1	5
u	0	2	4

$$= 2\int_0^2 u \cdot u\,du + \int_2^4 u \cdot u\,du - \left[\left(\frac{5^2}{2} - 5\right) - \left(\frac{(-1)^2}{2} - (-1)\right)\right]$$

$$= 2\int_0^2 u^2\,du + \int_2^4 u^2\,du - [6]$$

$$= 2\left[\frac{u^3}{3}\right]_0^2 + \left[\frac{u^3}{3}\right]_2^4 - 6$$

$$= \frac{2}{3}\left[2^3 - 0^3\right] + \frac{1}{3}\left[4^3 - 2^3\right] - 6$$

$$= \frac{2}{3}(8-0) + \frac{1}{3}(64-8) - 6$$

$$= \frac{16}{3} + \frac{56}{3} - 6$$

$$= \frac{72}{3} - 6 = 24 - 6 = 18.$$

Method ② ~ Over the y-axis. (i.e, integration with dy)

Enclosed Area =

$$\int_{-2}^{4} \left[(y+1) - \left(\frac{y^2}{2} - 3\right)\right] dy$$

$$= \int_{-2}^{4} \left(-\frac{y^2}{2} + y + 4\right) dy$$

$$= -\frac{1}{2}\left[\frac{y^3}{3}\right]_{-2}^{4} + \left[\frac{y^2}{2}\right]_{-2}^{4} + [4y]_{-2}^{4}$$

$$= 18$$